『经典参数估计与融合估计算法』

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0. Outline

考虑量测方程

$$\boxed{\boldsymbol{z} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{v}} \tag{1}$$

其中，$\boldsymbol{z} \in \mathbb{R}^N$, $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{H} \in \mathbb{R}^{N \times n}$, $\boldsymbol{v} \in \mathbb{R}^N \sim N(0, \boldsymbol{R})$. 此外，$\boldsymbol{R} \in \mathbb{R}^{N \times N}$.

我们的问题是基于量测 $\boldsymbol{z}$，对未知量 $\boldsymbol{x}$ 进行估计.

1. 最小二乘估计

取权重矩阵为 $\boldsymbol{W}=\boldsymbol{R}^{-1}$，（加权）最小二乘估计意味着

$$J=\frac{1}{2}||\boldsymbol{z}-\boldsymbol{H}\boldsymbol{x}||^{\boldsymbol{2}}_{\boldsymbol{R}^{-1}} \\ =\frac{1}{2}(\boldsymbol{z}-\boldsymbol{H}\boldsymbol{x})^{\boldsymbol{T}}\boldsymbol{R}^{-1}(\boldsymbol{z}-\boldsymbol{H}\boldsymbol{x})\Rightarrow min \tag{2}$$

由此可得

$$\hat{\boldsymbol{x}}_{\boldsymbol{LMS}}=(\boldsymbol{H}^{\boldsymbol{T}}\boldsymbol{R}^{-1}\boldsymbol{H})^{-1}\boldsymbol{H}^{\boldsymbol{T}}\boldsymbol{R}^{-1}\boldsymbol{z}$$

估计误差为

$$\begin{aligned}\tilde{\boldsymbol{x}}_{\boldsymbol{LMS}} &= \hat{\boldsymbol{x}}_{\boldsymbol{LMS}} - \boldsymbol{x} \\ &= (\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{R}^{-1}(\boldsymbol{H}\boldsymbol{x}+\boldsymbol{v}) - \boldsymbol{x} \\ &= (\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{v}\end{aligned}$$

根据量测噪声 $\boldsymbol{v}$ 的性质，易知

$$E\tilde{\boldsymbol{x}}_{\boldsymbol{LMS}} = 0$$

$$\boldsymbol{P}_{\tilde{\boldsymbol{x}}_{\boldsymbol{LMS}}} = E\tilde{\boldsymbol{x}}_{\boldsymbol{LMS}}\tilde{\boldsymbol{x}}_{\boldsymbol{LMS}}^T = (\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H})^{-1}$$

对于确定性未知量 $\boldsymbol{x}$，有

$$
\begin{aligned}
E\hat{\boldsymbol{x}}_{\boldsymbol{LMS}} &= \boldsymbol{x} \\
\boldsymbol{P}_{\hat{\boldsymbol{x}}_{\boldsymbol{LMS}}} &= E(\hat{\boldsymbol{x}}_{\boldsymbol{LMS}} - \boldsymbol{x})(\hat{\boldsymbol{x}}_{\boldsymbol{LMS}} - \boldsymbol{x})^{\boldsymbol{T}} \\
&= (\boldsymbol{H}^{\boldsymbol{T}}\boldsymbol{R}^{-1}\boldsymbol{H})^{-1} = \boldsymbol{P}_{\tilde{\boldsymbol{x}}_{\boldsymbol{LMS}}}
\end{aligned}
$$

所以，最小二乘估计可以表示为

$$
\boxed{\begin{cases}
\hat{\boldsymbol{x}}_{\boldsymbol{LMS}} = (\boldsymbol{H}^{\boldsymbol{T}}\boldsymbol{R}^{-1}\boldsymbol{H})^{-1}\boldsymbol{H}^{\boldsymbol{T}}\boldsymbol{R}^{-1}\boldsymbol{z} \\
\boldsymbol{P}_{\tilde{\boldsymbol{x}}_{\boldsymbol{LMS}}} = (\boldsymbol{H}^{\boldsymbol{T}}\boldsymbol{R}^{-1}\boldsymbol{H})^{-1}
\end{cases}} \tag{3}
$$

2. 极大似然估计

极大似然估计是指

$$f_{z|x}(\boldsymbol{z}|\boldsymbol{x}) \Rightarrow max \tag{4}$$

或

$$\ln f_{z|x}(\boldsymbol{z}|\boldsymbol{x}) \Rightarrow max \tag{5}$$

对于系统 (1)，注意到给定 $\boldsymbol{x}$ 时 $\boldsymbol{z} \sim N(\boldsymbol{H}\boldsymbol{x}, \boldsymbol{R})$，上述极值问题变为

$$J = \frac{1}{2}(\boldsymbol{z} - \boldsymbol{H}\boldsymbol{x})^{\boldsymbol{T}}\boldsymbol{R}^{-1}(\boldsymbol{z} - \boldsymbol{H}\boldsymbol{x}) \Rightarrow min \tag{6}$$

由此可得

$$\boxed{\begin{cases} \hat{\boldsymbol{x}}_{ML} = (\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{z} \\ \boldsymbol{P}_{\tilde{\boldsymbol{x}}_{ML}} = (\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H})^{-1} \end{cases}} \tag{7}$$

不难发现，和最小二乘估计是一致的。

3. 极大验后估计

极大验后估计是指

$$\ln f_{x|z}(\boldsymbol{x}|\boldsymbol{z}) \Rightarrow max \tag{8}$$

根据贝叶斯公式，有

$$f_{x|z}(\boldsymbol{x}|\boldsymbol{z}) = \frac{f_{z|x}(\boldsymbol{z}|\boldsymbol{x}) f_x(\boldsymbol{x})}{f_z(\boldsymbol{z})}$$

如果 $\boldsymbol{x} \sim N(\bar{\boldsymbol{x}}, \boldsymbol{P}_{\boldsymbol{x}})$，而且与 $\boldsymbol{v}$ 无关，极大似然估计意指

$$J = \frac{1}{2}(\boldsymbol{z} - \boldsymbol{H}\boldsymbol{x})^T \boldsymbol{R}^{-1}(\boldsymbol{z} - \boldsymbol{H}\boldsymbol{x}) + \frac{1}{2}(\boldsymbol{x} - \bar{\boldsymbol{x}})^T \boldsymbol{P}_{\boldsymbol{x}}^{-1}(\boldsymbol{x} - \bar{\boldsymbol{x}}) \Rightarrow min \tag{9}$$

由 $\frac{\partial J}{\partial \boldsymbol{x}}\big|_{\hat{x}} = \mathbf{0}$，可得

$$-\boldsymbol{H}^T\boldsymbol{R}^{-1}(\boldsymbol{z} - \boldsymbol{H}\hat{\boldsymbol{x}}) + \boldsymbol{P}_{\boldsymbol{x}}^{-1}(\hat{\boldsymbol{x}} - \bar{\boldsymbol{x}}) = \mathbf{0}$$

$$\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H}\hat{\boldsymbol{x}} + \boldsymbol{P}_{\boldsymbol{x}}^{-1}\hat{\boldsymbol{x}} = \boldsymbol{P}_{\boldsymbol{x}}^{-1}\bar{\boldsymbol{x}} + \boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{z}$$

即

$$\hat{\boldsymbol{x}}_{MA} = (\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H} + \boldsymbol{P}_{\boldsymbol{x}}^{-1})^{-1}(\boldsymbol{P}_{\boldsymbol{x}}^{-1}\bar{\boldsymbol{x}} + \boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{z}) \tag{10}$$

同时可以容易验证估计的无偏性，$E\hat{\boldsymbol{x}}_{MA}=\bar{\boldsymbol{x}}$. 此外

$$
\begin{aligned}
\tilde{\boldsymbol{x}}_{MA} &= \hat{\boldsymbol{x}}_{MA}-\boldsymbol{x}\\
&= (\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H}+\boldsymbol{P}_{\boldsymbol{x}}^{-1})^{-1}\\
&\times[\boldsymbol{P}_{\boldsymbol{x}}^{-1}\bar{\boldsymbol{x}}+\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{z}-(\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H}+\boldsymbol{P}_{\boldsymbol{x}}^{-1})\boldsymbol{x}]\\
&= (\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H}+\boldsymbol{P}_{\boldsymbol{x}}^{-1})^{-1}[\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{v}-\boldsymbol{P}_{\boldsymbol{x}}^{-1}\mathring{\boldsymbol{x}}]
\end{aligned}
$$

由此，可得

$$
\begin{aligned}
\boldsymbol{P}_{\tilde{\boldsymbol{x}}_{MA}} &= E\tilde{\boldsymbol{x}}_{MA}\tilde{\boldsymbol{x}}_{MA}^T\\
&= (\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H}+\boldsymbol{P}_{\boldsymbol{x}}^{-1})^{-1}[\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H}+\boldsymbol{P}_{\boldsymbol{x}}^{-1}](\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H}+\boldsymbol{P}_{\boldsymbol{x}}^{-1})^{-1}\\
&= (\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H}+\boldsymbol{P}_{\boldsymbol{x}}^{-1})^{-1}
\end{aligned}
$$

总结起来，极大验后估计即为

$$\boxed{\begin{cases} \hat{\boldsymbol{x}}_{MA} = \boldsymbol{P}_{\tilde{\boldsymbol{x}}_{MA}}(\boldsymbol{P}_{\boldsymbol{x}}^{-1}\bar{\boldsymbol{x}} + \boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{z}) \\ \boldsymbol{P}_{\tilde{\boldsymbol{x}}_{MA}} = (\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H} + \boldsymbol{P}_{\boldsymbol{x}}^{-1})^{-1} \end{cases}} \tag{11}$$

当无先验信息时，$\boldsymbol{P}_{\boldsymbol{x}}^{-1} = \mathbf{0}$. 可见，此时极大验后估计与最小二乘估计、极大似然估计都是一致的。

4. 最小方差估计

对于系统 (1)，最小方差估计即为线性最小方差估计。假设 $\boldsymbol{x} \sim N(\bar{\boldsymbol{x}}, \boldsymbol{P}_{\boldsymbol{x}})$，且与 $\boldsymbol{v} \sim N(\boldsymbol{0}, \boldsymbol{R})$ 无关. 因为 $\hat{\boldsymbol{x}}_{MV} = \bar{\boldsymbol{x}} + \boldsymbol{P}_{\boldsymbol{xz}}\boldsymbol{P}_{\boldsymbol{z}}^{-1}(\boldsymbol{z} - \bar{\boldsymbol{z}})$，注

意到

$$
\begin{aligned}
\bar{\boldsymbol{z}} &= \boldsymbol{H}\bar{\boldsymbol{x}} \\
\boldsymbol{P}_z &= E\mathring{\boldsymbol{z}}\mathring{\boldsymbol{z}}^{\boldsymbol{T}} = E(\boldsymbol{H}\mathring{\boldsymbol{x}} + \boldsymbol{v})(\boldsymbol{H}\mathring{\boldsymbol{x}} + \boldsymbol{v})^{\boldsymbol{T}} \\
&= \boldsymbol{H}\boldsymbol{P}_x\boldsymbol{H}^{\boldsymbol{T}} + \boldsymbol{R} \\
\boldsymbol{P}_{xz} &= E\mathring{\boldsymbol{x}}\mathring{\boldsymbol{z}}^{\boldsymbol{T}} = E\mathring{\boldsymbol{x}}(\boldsymbol{H}\mathring{\boldsymbol{x}} + \boldsymbol{v})^{\boldsymbol{T}} \\
&= \boldsymbol{P}_x\boldsymbol{H}^{\boldsymbol{T}}
\end{aligned}
$$

所以

$$
\hat{\boldsymbol{x}}_{MV} = \bar{\boldsymbol{x}} + \boldsymbol{P}_x\boldsymbol{H}^{\boldsymbol{T}}(\boldsymbol{H}\boldsymbol{P}_x\boldsymbol{H}^{\boldsymbol{T}} + \boldsymbol{R})^{-1}(\boldsymbol{z} - \boldsymbol{H}\bar{\boldsymbol{x}}) \tag{12}
$$

$$\begin{aligned} P_{\tilde{x}_{MV}} &= P_x - P_{xz} P_z^{-1} P_{zx} \\ &= P_x - P_x H^T (H P_x H^T + R)^{-1} H P_x \end{aligned} \tag{13}$$

由矩阵求逆引理（Matrix Inversion Lemma）

$$(A + BCD)^{-1} = A^{-1} - A^{-1} B (D A^{-1} B + C^{-1})^{-1} D A^{-1} \tag{14}$$

对照取 $A = P_x^{-1}$, $B = H^T$, $D = H$, $C = R^{-1}$, 式（13）可化为

$$P_{\tilde{x}_{MV}} = (P_x^{-1} + H^T R^{-1} H)^{-1} \tag{15}$$

另外，式 (12) 可以化为

$$\hat{x}_{MV} = \underline{[I - P_x H^T (H P_x H^T + R)^{-1} H]} \bar{x} + \underline{P_x H^T (H P_x H^T + R)^{-1}} z$$

注意到

$$
\begin{aligned}
&[\boldsymbol{I}-\boldsymbol{P}_x\boldsymbol{H}^T(\boldsymbol{H}\boldsymbol{P}_x\boldsymbol{H}^T+\boldsymbol{R})^{-1}\boldsymbol{H}] \\
&\qquad =[\boldsymbol{P}_x-\boldsymbol{P}_x\boldsymbol{H}^T(\boldsymbol{H}\boldsymbol{P}_x\boldsymbol{H}^T+\boldsymbol{R})^{-1}\boldsymbol{H}\boldsymbol{P}_x]\boldsymbol{P}_x^{-1} \\
&\qquad =\boldsymbol{P}_{\tilde{x}_{MV}}\boldsymbol{P}_x^{-1} \\
&\qquad =(\boldsymbol{P}_x^{-1}+\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H})^{-1}\boldsymbol{P}_x^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\boldsymbol{P}_x\boldsymbol{H}^T(\boldsymbol{H}\boldsymbol{P}_x\boldsymbol{H}^T+\boldsymbol{R})^{-1} \\
&\quad =(**)^{-1}\boxed{\underbrace{(\boldsymbol{P}_x^{-1}+\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H})}_{**}\boldsymbol{P}_x\boldsymbol{H}^T}(\boldsymbol{H}\boldsymbol{P}_x\boldsymbol{H}^T+\boldsymbol{R})^{-1} \\
&\quad =(\boldsymbol{P}_x^{-1}+\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{R}^{-1}
\end{aligned}
$$

因此，有

$$\hat{\boldsymbol{x}}_{MV} = (\boldsymbol{P}_{\boldsymbol{x}}^{-1} + \boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H})^{-1}(\boldsymbol{P}_{\boldsymbol{x}}^{-1}\bar{\boldsymbol{x}} + \boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{z}) = \hat{\boldsymbol{x}}_{MA} \tag{16}$$

上述讨论表明，对于系统 (1), 最小方差估计与极大验后估计是一致的.

5. 融合估计

对于未知的 $\boldsymbol{x}$，设独立地获得了两个估计：$(\hat{\boldsymbol{x}}_1, \boldsymbol{P}_1)$ 和 $(\hat{\boldsymbol{x}}_2, \boldsymbol{P}_2)$. 可以认为

$$\boxed{\begin{aligned} \hat{\boldsymbol{x}}_1 &= \boldsymbol{x} + \boldsymbol{v}_1, \qquad & \boldsymbol{v}_1 &\sim \boldsymbol{N}(0, \boldsymbol{P}_1) \\ \hat{\boldsymbol{x}}_2 &= \boldsymbol{x} + \boldsymbol{v}_2, \qquad & \boldsymbol{v}_2 &\sim \boldsymbol{N}(0, \boldsymbol{P}_2) \end{aligned}}$$

即

$$\begin{bmatrix} \hat{\boldsymbol{x}}_1 \\ \hat{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_n \\ \boldsymbol{I}_n \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}$$

记

$$\mathcal{H} = \begin{bmatrix} \boldsymbol{I}_n \\ \boldsymbol{I}_n \end{bmatrix}, \mathcal{V} = \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}$$

那么

$$E\mathcal{V} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, E\mathcal{V}\mathcal{V}^T \triangleq \mathcal{R} = \begin{bmatrix} \boldsymbol{P}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P}_2 \end{bmatrix},$$

注意到

$$\mathcal{H}^T\mathcal{R}^{-1} = [\boldsymbol{I}_n, \boldsymbol{I}_n]\begin{bmatrix} \boldsymbol{P}_1^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_2^{-1} \end{bmatrix} = [\boldsymbol{P}_1^{-1}, \boldsymbol{P}_2^{-1}]$$

$$\mathcal{H}^T\mathcal{R}^{-1}\begin{bmatrix} \hat{\boldsymbol{x}}_1 \\ \hat{\boldsymbol{x}}_2 \end{bmatrix} = \boldsymbol{P}_1^{-1}\hat{\boldsymbol{x}}_1 + \boldsymbol{P}_2^{-1}\hat{\boldsymbol{x}}_2$$

$$(\mathcal{H}^T\mathcal{R}^{-1}\mathcal{H})^{-1} = (\boldsymbol{P}_1^{-1} + \boldsymbol{P}_2^{-1})^{-1}$$

由最小二乘估计可知

$$\hat{\boldsymbol{x}} = (\boldsymbol{P}_1^{-1} + \boldsymbol{P}_2^{-1})^{-1}(\boldsymbol{P}_1^{-1}\hat{\boldsymbol{x}}_1 + \boldsymbol{P}_2^{-1}\hat{\boldsymbol{x}}_2)$$

$$\boldsymbol{P} = E(\hat{\boldsymbol{x}}_1 - \boldsymbol{x})(\hat{\boldsymbol{x}}_1 - \boldsymbol{x})^T = (\boldsymbol{P}_1^{-1} + \boldsymbol{P}_2^{-1})^{-1}$$

将上述结论可以总结为如下定理.

Theorem 5.1 如果对于 x 有两个独立的最优估计 $(\hat{x}_1, P_1)$ 和 $(\hat{x}_2, P_2)$，那么融合后的最优估计为

$$\boxed{\begin{cases} P^{-1} = P_1^{-1} + P_2^{-1} \\ \hat{x} = P(P_1^{-1}\hat{x}_1 + P_2^{-1}\hat{x}_2) \end{cases}} \tag{17}$$

Example 5.1 已知 $(\hat{x}_1 = 1, P_1 = 0.1)$，$(\hat{x}_2 = 2, P_2 = 0.5)$，求融合估计 $\hat{x}$.

【解】因为 $P^{-1}=P_1^{-1}+P_2^{-1}=10+2=12$，所以

$$\hat{x}=P(P_1^{-1}(\hat{x}_1+P_2^{-1}\hat{x}_2)=\frac{1}{12}(10\hat{x}_1+2\hat{x}_2)$$
$$=\frac{10}{12}+\frac{4}{12}=\frac{14}{12}\approx 1.167$$
$$P=\frac{1}{12}\approx 0.083$$

□

进一步，我们可以建立如下更一般的结论。

Theorem 5.2 如果对于 x 有 N 个相互独立的最优估计 $(\hat{\boldsymbol{x}}_1, \boldsymbol{P}_1)$, $(\hat{\boldsymbol{x}}_2, \boldsymbol{P}_2)$, $\cdots$, $(\hat{\boldsymbol{x}}_N, \boldsymbol{P}_N)$. 那么，融合后的最优估计为

$$\boxed{\begin{cases} \boldsymbol{P}^{-1} = \sum_{i=1}^{N} \boldsymbol{P}_i^{-1} \\ \hat{\boldsymbol{x}} = \boldsymbol{P} \sum_{i=1}^{N} \boldsymbol{P}_i^{-1} \hat{\boldsymbol{x}}_i \end{cases}} \tag{18}$$

算法 (18) 是集中式融合算法. 可以容易地建立如下序贯式融合算法:

$$\boxed{\begin{cases} \boldsymbol{P}_{(k+1)}^{-1} = \boldsymbol{P}_{(k)}^{-1} + \boldsymbol{P}_{k+1}^{-1} \\ \hat{\boldsymbol{x}}_{(k+1)} = \boldsymbol{P}_{(k+1)} [\boldsymbol{P}_{(k)}^{-1} \hat{\boldsymbol{x}}_{(k)} + \boldsymbol{P}_{k+1}^{-1} \hat{\boldsymbol{x}}_{k+1}] \end{cases}} \tag{19}$$

其中，$(\hat{\boldsymbol{x}}_{(\boldsymbol{k})}, \boldsymbol{P}_{(\boldsymbol{k})}^{-1})$ 表示前 k 个传感器的融合估计结果，$(\hat{\boldsymbol{x}}_{k+1}, \boldsymbol{P}_{k+1}^{-1})$ 表示第 $k+1$ 个传感器的最优估计，$(\hat{\boldsymbol{x}}_{(\boldsymbol{k+1})}, \boldsymbol{P}_{(\boldsymbol{k+1})}^{-1})$ 表示 $k+1$ 个传感器的融合估计结果.

5.1 融合估计的最优性

对于估计问题

$$\begin{cases} \boldsymbol{z} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{v}, & \boldsymbol{v} \sim N(0, \boldsymbol{R}) \\ \boldsymbol{x} \sim N(\bar{\boldsymbol{x}}, \boldsymbol{P}_{\boldsymbol{x}}) \end{cases}$$

相当于

$$\begin{cases} \boldsymbol{z} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{v}, & \boldsymbol{v} \sim N(0, \boldsymbol{R}) \\ \bar{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{\epsilon}, & \boldsymbol{\epsilon} \sim N(\boldsymbol{0}, \boldsymbol{P}_{\boldsymbol{x}}) \end{cases}$$

可见，融合估计对应的优化问题为

$$J = \frac{1}{2}(\boldsymbol{z} - \boldsymbol{H}\boldsymbol{x})^{\boldsymbol{T}}\boldsymbol{R}^{-1}(\boldsymbol{z} - \boldsymbol{H}\boldsymbol{x}) + \frac{1}{2}(\boldsymbol{x} - \bar{\boldsymbol{x}})^{\boldsymbol{T}}\boldsymbol{P}_{\boldsymbol{x}}^{-1}(\boldsymbol{x} - \bar{\boldsymbol{x}}) \Rightarrow min \quad (20)$$

这与极大验后估计是一致的。

5.2 融合估计的等价解算方法

对于未知的 $\boldsymbol{x}$，基于两个独立估计 $(\hat{\boldsymbol{x}}_1, \boldsymbol{P}_1)$ 和 $(\hat{\boldsymbol{x}}_2, \boldsymbol{P}_2)$ 的综合估计假设为

$$\hat{\boldsymbol{x}} = \boldsymbol{K}_1\hat{\boldsymbol{x}}_1 + \boldsymbol{K}_2\hat{\boldsymbol{x}}_2 \tag{21}$$

式中，$\boldsymbol{K}_1, \boldsymbol{K}_2$ 是两个待定系数矩阵. 根据估计的无偏性要求，可知

$$\boldsymbol{K}_1 + \boldsymbol{K}_2 = \boldsymbol{I} \tag{22}$$

记 $\boldsymbol{K}_1 = \boldsymbol{K}$，那么 $\boldsymbol{K}_2 = \boldsymbol{I} - \boldsymbol{K}$. 因此

$$\hat{\boldsymbol{x}} = \boldsymbol{K}\hat{\boldsymbol{x}}_1 + (\boldsymbol{I} - \boldsymbol{K})\hat{\boldsymbol{x}}_2$$

$$\tilde{\boldsymbol{x}} = \hat{\boldsymbol{x}} - \boldsymbol{x} = \boldsymbol{K}\tilde{\boldsymbol{x}}_1 + (\boldsymbol{I} - \boldsymbol{K})\tilde{\boldsymbol{x}}_2$$

由此可得估计误差指标函数

$$\begin{aligned} J &= E||\tilde{\boldsymbol{x}}||^2 = tr(E\tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}^{\boldsymbol{T}}) \\ &= tr[\boldsymbol{K}\boldsymbol{P}_1\boldsymbol{K}^{\boldsymbol{T}} + (\boldsymbol{I} - \boldsymbol{K})\boldsymbol{P}(\boldsymbol{I} - \boldsymbol{K})^{\boldsymbol{T}}] \end{aligned}$$

注意到 $\frac{\partial}{\partial \boldsymbol{A}} tr(\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^{\boldsymbol{T}}) = \boldsymbol{A}\boldsymbol{B}^{\boldsymbol{T}} + \boldsymbol{A}\boldsymbol{B}$. 当 $\boldsymbol{B}^{\boldsymbol{T}} = \boldsymbol{B}$ 时，$\frac{\partial}{\partial \boldsymbol{A}} tr(\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^{\boldsymbol{T}}) = 2\boldsymbol{A}\boldsymbol{B}$. 由 $\frac{\partial J}{\partial \boldsymbol{K}} = 0$，可导出

$$\boldsymbol{K}\boldsymbol{P}_1 - (\boldsymbol{I} - \boldsymbol{K})\boldsymbol{P}_2 = \boldsymbol{0}$$

$$\boldsymbol{K}(\boldsymbol{P}_1 + \boldsymbol{P}_2) = \boldsymbol{P}_2$$

因此

$$\boldsymbol{K} = \boldsymbol{P}_2(\boldsymbol{P}_1 + \boldsymbol{P}_2)^{-1} = (\boldsymbol{P}_1^{-1} + \boldsymbol{P}_2^{-1})^{-1}\boldsymbol{P}_1^{-1}$$

$$\boldsymbol{I} - \boldsymbol{K} = \boldsymbol{I} - (\boldsymbol{P}_1^{-1} + \boldsymbol{P}_2^{-1})^{-1}\boldsymbol{P}_1^{-1} = (\boldsymbol{P}_1^{-1} + \boldsymbol{P}_2^{-1})^{-1}\boldsymbol{P}_2^{-1}$$

因此

$$\hat{\boldsymbol{x}} = (\boldsymbol{P}_1^{-1} + \boldsymbol{P}_2^{-1})^{-1}(\boldsymbol{P}_1^{-1}\hat{\boldsymbol{x}}_1 + \boldsymbol{P}_2^{-1}\hat{\boldsymbol{x}}_2)$$

$$\boldsymbol{P} = \boldsymbol{P}_{\tilde{\boldsymbol{x}}} = (\boldsymbol{P}_1^{-1} + \boldsymbol{P}_2^{-1})^{-1}(\boldsymbol{P}_1^{-1} + \boldsymbol{P}_2^{-1})(\boldsymbol{P}_1^{-1} + \boldsymbol{P}_2^{-1})^{-1}$$

即

$$\boxed{\begin{cases} \hat{\boldsymbol{x}} = \boldsymbol{P}(\boldsymbol{P}_1^{-1}\hat{\boldsymbol{x}}_1 + \boldsymbol{P}_2^{-1}\hat{\boldsymbol{x}}_2) \\ \boldsymbol{P}^{-1} = \boldsymbol{P}_1^{-1} + \boldsymbol{P}_2^{-1} \end{cases}} \tag{23}$$

5.3 融合估计与最小方差估计

对于参数估计问题

$$\boldsymbol{z} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{v}, \qquad \boldsymbol{v} \sim N(\boldsymbol{0}, \boldsymbol{R})$$

我们首先可以容易地获得最小二乘估计（或极大似然估计）

$$\begin{cases} \hat{\boldsymbol{x}}_1 = (\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{z} \\ \boldsymbol{P}_1 = (\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H})^{-1} \end{cases} \tag{24}$$

如果我们有关于 $\boldsymbol{x}$ 的先验知识，即 $\boldsymbol{x} \sim N(\bar{\boldsymbol{x}}, \boldsymbol{P}_{\boldsymbol{x}})$，可以认为是获得了如下估计：

$$\begin{cases} \hat{\boldsymbol{x}}_2 = \bar{\boldsymbol{x}} \\ \boldsymbol{P}_2 = \boldsymbol{P}_{\boldsymbol{x}} \end{cases} \tag{25}$$

根据融合估计定理5.1，将估计（24）和（25）融合可得

$$\begin{cases} \boldsymbol{P}^{-1} = \boldsymbol{P}_{\boldsymbol{x}}^{-1} + (\boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H})^{-1} \\ \hat{\boldsymbol{x}} = \boldsymbol{P}(\boldsymbol{P}_{\boldsymbol{x}}^{-1}\bar{\boldsymbol{x}} + \boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{z}) \end{cases} \tag{26}$$

注意到最小方差估计（15）、（16），即

$$\begin{cases} \boldsymbol{P}_{\tilde{\boldsymbol{x}}_{MV}} = (\boldsymbol{P}_{\boldsymbol{x}}^{-1} + \boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H})^{-1} \\ \hat{\boldsymbol{x}}_{MV} = (\boldsymbol{P}_{\boldsymbol{x}}^{-1} + \boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{H})^{-1}(\boldsymbol{P}_{\boldsymbol{x}}^{-1}\bar{\boldsymbol{x}} + \boldsymbol{H}^T\boldsymbol{R}^{-1}\boldsymbol{z}) \end{cases} \tag{27}$$

比较（26）和（27），不难发现两者是一致的，这进一步说明了上面介绍的融合估计算法的最优性.

5.4 递推最小二乘估计

由 $k+1$ 时刻的量测

$$\boldsymbol{z}_{k+1} = \boldsymbol{H}_{k+1}\boldsymbol{x} + \boldsymbol{v}_{k+1} \tag{28}$$

可以获得附加的估计

$$\begin{cases} \hat{\boldsymbol{x}}_a = (\boldsymbol{H}_{k+1}^T \boldsymbol{w}_{k+1} \boldsymbol{H}_{k+1})^{-1} \boldsymbol{H}_{k+1}^T \boldsymbol{w}_{k+1} \boldsymbol{z}_{k+1} \\ \boldsymbol{P}_a = (\boldsymbol{H}_{k+1}^T \boldsymbol{w}_{k+1} \boldsymbol{H}_{k+1})^{-1} \end{cases} \tag{29}$$

与已经获得被估计量 x 在 k 时刻的估计 $(\hat{\boldsymbol{x}}_k, \boldsymbol{P}_k)$ 进行融合，得

$$\begin{cases} \hat{\boldsymbol{x}}_{k+1} = \boldsymbol{P}_{k+1}(\boldsymbol{P}_k^{-1}\hat{\boldsymbol{x}}_k + \boldsymbol{P}_a^{-1}\hat{\boldsymbol{x}}_a) \\ \boldsymbol{P}_{k+1} = (\boldsymbol{P}_k^{-1} + \boldsymbol{P}_a^{-1})^{-1} \end{cases} \tag{30}$$

因而有

$$\boldsymbol{P}_{k+1} = (\boldsymbol{P}_k^{-1} + \boldsymbol{H}_{k+1}^T \boldsymbol{w}_{k+1} \boldsymbol{H}_{k+1})^{-1} \tag{31}$$

同时

$$\begin{aligned} \hat{\boldsymbol{x}}_{k+1} &= \boldsymbol{P}_{k+1}[(\boldsymbol{P}_k^{-1} + \boldsymbol{P}_a^{-1})\hat{\boldsymbol{x}}_k + \boldsymbol{P}_a^{-1}\hat{\boldsymbol{x}}_a - \boldsymbol{P}_a^{-1}\hat{\boldsymbol{x}}_k] \\ &= \hat{\boldsymbol{x}}_k + \boldsymbol{P}_{k+1}\boldsymbol{H}_{k+1}^T\boldsymbol{w}_{k+1}\boldsymbol{z}_{k+1} - \boldsymbol{P}_{k+1}\boldsymbol{H}_{k+1}^T\boldsymbol{w}_{k+1}\boldsymbol{H}_{k+1}\hat{\boldsymbol{x}}_k \end{aligned}$$

即

$$\hat{\boldsymbol{x}}_{k+1} = \hat{\boldsymbol{x}}_{\boldsymbol{k}} + \boldsymbol{P}_{\boldsymbol{k+1}} \boldsymbol{H}_{\boldsymbol{k+1}}^{\boldsymbol{T}} \boldsymbol{w}_{\boldsymbol{k+1}} (\boldsymbol{z}_{\boldsymbol{k+1}} - \boldsymbol{H}_{\boldsymbol{k+1}} \hat{\boldsymbol{x}}_{\boldsymbol{k}}) \tag{32}$$

式 (31) 和 (32) 便构成了递推最小二乘估计，即

$$\boxed{\begin{cases} \hat{\boldsymbol{x}}_{k+1} = \hat{\boldsymbol{x}}_{\boldsymbol{k}} + \boldsymbol{P}_{\boldsymbol{k+1}} \boldsymbol{H}_{\boldsymbol{k+1}}^{\boldsymbol{T}} \boldsymbol{w}_{\boldsymbol{k+1}} (\boldsymbol{z}_{\boldsymbol{k+1}} - \boldsymbol{H}_{\boldsymbol{k+1}} \hat{\boldsymbol{x}}_{\boldsymbol{k}}) \\ \boldsymbol{P}_{k+1} = (\boldsymbol{P}_{\boldsymbol{k}}^{\boldsymbol{-1}} + \boldsymbol{H}_{\boldsymbol{k+1}}^{\boldsymbol{T}} \boldsymbol{w}_{\boldsymbol{k+1}} \boldsymbol{H}_{\boldsymbol{k+1}})^{-1} \end{cases}} \tag{33}$$

以上讨论表明基于融合估计的思想，建立递推最小二乘估计比传统方法要简明、容易得多. □