『线性最优滤波理论』

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0. Outline

1. 离散时间随机系统基本分析

1.1 数学描述

考虑如下系统:

$$x_{k+1} = \Phi_{k+1,k}x_k + \Psi_{k+1,k}u_k + \Gamma_k w_k \quad (1)$$

$$y_{k+1} = H_{k+1}x_{k+1} + v_{k+1} \quad (2)$$

其中:

- u_k 为控制信号，是确定性输入;
- w_k 称为系统的过程噪声，有时又称为模型噪声，假设为均值为零的高斯不相关噪声序列;

- v_{k+1} 称为系统的量测噪声，同样假设它是均值为零的高斯不相关噪声序列.

1.2 基本假设

关于过程噪声

$$
\begin{aligned}
E[w_k] &= 0 \\
\operatorname{cov}[w_k, w_j] &= Q_k \delta_{kj}
\end{aligned}
$$

其中，$Q_k \geq 0$;

关于量测噪声

$$
\begin{aligned}
E[v_k] &= 0 \\
\operatorname{cov}[v_k, v_j] &= R_k \delta_{kj}
\end{aligned}
$$

其中，$R_k > 0$；

假设系统的初始状态 x_0 也是高斯分布的随机矢量，即 $x_0 \sim N(m_0, P_0)$.

$$Ex_0 = m_0 \tag{3}$$
$$\text{cov}[x_0] = P_0 \tag{4}$$

进一步还假设

$$\text{cov}[w_k, v_{j+1}] = 0, \quad \forall k, j \geq 0 \tag{5}$$
$$\text{cov}[w_k, x_0] = 0, \quad \forall k \geq 0 \tag{6}$$
$$\text{cov}[v_{k+1}, x_0] = 0, \quad \forall k \geq 0 \tag{7}$$

1.3 统计分析

考虑到（为书写方便，下式中暂时认为 $u_k = 0$）

$$
\begin{aligned}
x_k &= \Phi_{k,k-1}x_{k-1} + \Gamma_{k-1}w_{k-1} \\
&= \Phi_{k,k-1}\Phi_{k-1,k-2}x_{k-2} + \Phi_{k,k-1}\Gamma_{k-2}w_{k-2} + \Gamma_{k-1}w_{k-1} \\
&\vdots \\
&= \Phi_{k,0}x_0 + \sum_{i=1}^{k}\Phi_{k,i}\Gamma_{i-1}w_{i-1}
\end{aligned}
$$

其中，$\Phi_{k,0} = \Phi_{k,k-1}\Phi_{k-1,k-2}\cdots\Phi_{1,0}$. 而

$$
y_k = H_k x_k + v_k
$$

由此可知，x_k 与 y_k 也是高斯随机矢量.

当 $k < j$ 时，有

$$E < x_k, w_j >= 0 \tag{8}$$

$$E < y_k, v_j >= 0 \tag{9}$$

状态均值传播方程

$$\overline{x}_{k+1} = \Phi_{k+1,k}\overline{x}_k + \Psi_{k+1,k}u_k \tag{10}$$

$$\overline{x}_0 = m_0 \tag{11}$$

状态方差传播方程

因为

$$x_{k+1} - \overline{x}_{k+1} = \Phi_{k+1,k}[x_k - \overline{x}_k)] + \Gamma_k[w_k - \overline{w}_k]$$

所以

$$\begin{aligned} P_{k+1} &= E[x_{k+1} - \overline{x}_{k+1}][x_{k+1} - \overline{x}_{k+1}]^{\mathrm{T}} \\ &= \Phi_{k+1,k} P_k \Phi_{k+1,k}^{\mathrm{T}} + \Gamma_k Q_k \Gamma_k^{\mathrm{T}} \\ &\quad + \Phi_{k+1,k} \overline{[x_k - \overline{x}_k][w_k - \overline{w}_k]^{\mathrm{T}}} \Gamma_k^{\mathrm{T}} \\ &\quad + \Gamma_k \overline{[w_k - \overline{w}_k][x_k - \overline{x}_k]^{\mathrm{T}}} \Phi_{k+1,k}^{\mathrm{T}} \end{aligned}$$

考虑到 (8)，最后可得

$$P_{k+1} = \Phi_{k+1,k} P_k \Phi_{k+1,k}^{\mathrm{T}} + \Gamma_k Q_k \Gamma_k^{\mathrm{T}} \tag{12}$$

记

$$P_{k,j} = \mathrm{cov}[x_k, x_j]$$

当 $k \geq j$ 时，因为

$$x_k = \Phi_{k,j} x_j + \sum_{i=j+1}^{k} \Phi_{k,i} \Gamma_{i-1} w_{i-1}$$

$$\Rightarrow \quad \begin{aligned} \bar{x}_k &= \Phi_{k,j} \bar{x}_j \\ \mathring{x}_k &= \Phi_{k,j} \mathring{x}_j + \textstyle\sum_{i=j+1}^{k} \Phi_{k,i} \Gamma_{i-1} w_{i-1} \\ P_{k,j} &= E \mathring{x}_k \mathring{x}_j^T = \Phi_{k,j} E \mathring{x}_j \mathring{x}_j^T = \Phi_{k,j} P_{j,j} = \Phi_{k,j} P_j \end{aligned}$$

所以

$$P_{k,j} = \begin{cases} \Phi_{k,j} P_j & \text{if} \quad k \geq j \\ P_k \Phi_{j,k}^{\mathrm{T}} & \text{if} \quad k < j \end{cases} \tag{13}$$

2. 状态估计基本引理

根据前面介绍的静态估计理论，不难建立下述四个结论.

Lemma 2.1 基于量测序列 $\mathbf{y}_j \triangleq [y_1^{\mathrm{T}}, y_2^{\mathrm{T}}, \cdots, y_j^{\mathrm{T}}]^{\mathrm{T}}$ 对状态 x_k 的最小方差估计为条件均值

$$\hat{x}_{k|j} = E[x_k|\mathbf{y}_j]$$

此外，该估计是无偏的.

Lemma 2.2 若待估计量 x_k（简记为 x）与量测序列 $\mathbf{y}_j$（简记为 y）的联合分布是高斯的，那么

$$E(x|y) = \hat{x} = \overline{x} + P_{xy}P_y^{-1}(y - \overline{y})$$

式中，$P_{xy} = \text{cov}(x, y)$，$P_y = \text{var}(y)$.

注：如果没有高斯分布假设条件，我们有（线性最小方差估计）

$$\hat{x}_L = \bar{x} + P_{xy}P_y^{-1}(y - \bar{y})$$

Lemma 2.3 若对估计量 x 有两组相互独立的量测 y 与 z，而且 (x, y, z) 的联合分布是高斯的，那么

$$\hat{x} = E(x|y, z) = E(x|y) + E(x|z) - \overline{x}$$

注：线性最小方差估计有类似结论

$$\hat{x}_L(y,z)=\hat{x}_L(y)+\hat{x}_L(z)-\bar{x}$$

Lemma 2.4 在上述引理 3 中，若量测 y 与 z 相关，那么

$$\hat{x}=E(x|y,z)=E(x|y,\tilde{z})=E(x|y)+E(x|\tilde{z})-\bar{x}$$

其中

$$\tilde{z}=z-E(z|y)=z-\bar{z}-P_{zy}P_y^{-1}(y-\bar{y})$$

而且 $E\tilde{z}=0,\operatorname{cov}(\tilde{z},y)=0.$

注：对于线性最小方差估计，无需高斯分布假设，有

$$\hat{x}_L(y,z) = \hat{x}_L(y,\tilde{z}) = \hat{x}_L(y) + \hat{x}_L(\tilde{z}) - \bar{x}$$

引理 4 证明

记 $v = x - \bar{x} - P_{xy}P_y^{-1}[y-\bar{y}] - P_{x\tilde{z}}P_{\tilde{z}}^{-1}\tilde{z}$, 那么

$$\begin{aligned} Ev &= 0 \\ \mathrm{cov}[v,y] &= P_{xy} - P_{xy}P_y^{-1}P_y - P_{x\tilde{z}}P_{\tilde{z}}^{-1}P_{\tilde{z}y} = 0 \\ \mathrm{cov}[v,\tilde{z}] &= P_{x\tilde{z}} - P_{xy}P_y^{-1}P_{y\tilde{z}} - P_{x\tilde{z}}P_{\tilde{z}}^{-1}P_{\tilde{z}} = 0 \end{aligned}$$

由此可知 $\{v, y, \tilde{z}\}$ 三者是相互独立的高斯型随机量，现进行如下变换：

$$\begin{aligned} x &= v + \bar{x} + P_{xy}P_y^{-1}[y-\bar{y}] + P_{x\tilde{z}}P_{\tilde{z}}^{-1}\tilde{z} \\ z &= \bar{z} + P_{zy}P_y^{-1}[y-\bar{y}] + \tilde{z} \\ y &= y \end{aligned}$$

$$f(x,z,y)=f(v,\tilde{z},y)\left|\frac{\partial(x,z,y)}{\partial(v,\tilde{z},y)}\right|=f(v)f(\tilde{z})f(y)$$

其中

$$\left|\frac{\partial(x,z,y)}{\partial(v,\tilde{z},y)}\right|=\det\begin{bmatrix}\frac{\partial x}{\partial v} & \frac{\partial x}{\partial \tilde{z}} & \frac{\partial x}{\partial y}\\ \frac{\partial z}{\partial v} & \frac{\partial z}{\partial \tilde{z}} & \frac{\partial z}{\partial y}\\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial \tilde{z}} & \frac{\partial y}{\partial y}\end{bmatrix}=\det\begin{bmatrix} I & P_{x\tilde{z}}P_{\tilde{z}}^{-1} & P_{xy}P_y^{-1}\\ 0 & I & P_{zy}P_y^{-1}\\ 0 & 0 & I\end{bmatrix}=1$$

$$\begin{aligned}f(x|z,y) &= \frac{f(x,z,y)}{f(z,y)}=\frac{f(v,\tilde{z},y)}{f(z,y)}=\frac{f(v)f(\tilde{z})}{f(z|y)}\\ &= f(v)=\frac{1}{\sqrt{(2\pi)^n|P_v|}}\exp[-\frac{1}{2}v^{\mathrm{T}}P_v^{-1}v]\end{aligned}$$

其中, $v = x - \bar{x} - P_{xy}P_y^{-1}[y - \bar{y}] - P_{x\tilde{z}}P_{\tilde{z}}^{-1}\tilde{z}$.

从而可知:

$$\hat{x} = E[x|z, y] = \bar{x} + P_{xy}P_y^{-1}[y - \bar{y}] + P_{x\tilde{z}}P_{\tilde{z}}^{-1}\tilde{z}$$

即 $\hat{x} = E[x|y] + E[x|\tilde{z}] - \bar{x} = E[x|y, \tilde{z}]$. ■

3. 卡尔曼滤波算法

一般状态最优估计问题可以描述为

$$\hat{x}_{t|k} = E[x_t|y_1, y_2, \cdots, y_k] = E[x_t|\mathbf{y}_k]$$

可分为三类子问题:

- ◇ $t < k$: 平滑;
- ◇ $t = k$: 滤波;
- ◇ $t > k$: 预测.

3.1 一步预测（时间修正）

Time Updating

$$\hat{x}_{k+1|k} = E[x_{k+1}|\mathbf{y}_k] = E[\Phi_{k+1,k}x_k + \Psi_{k+1,k}u_k + \Gamma_k w_k|\mathbf{y}_k]$$

即

$$\boxed{\begin{aligned} \hat{x}_{k+1|k} &= \Phi_{k+1,k}\hat{x}_{k|k} + \Psi_{k+1,k}u_k \quad (14) \\ P_{k+1|k} &= E\tilde{x}_{k+1|k}\tilde{x}^{\mathrm{T}}_{k+1|k} = \Phi_{k+1,k}P_{k|k}\Phi^{\mathrm{T}}_{k+1,k} + \Gamma_k Q_k \Gamma^{\mathrm{T}}_k \quad (15) \end{aligned}}$$

以上两式表明，只要知道 $\hat{x}_{k|k}$ 和 $P_{k|k}$，便可求出 $\hat{x}_{k+1|k}$ 和 $P_{k+1|k}$.

3.2 新息（innovation, new information）

新息定义为

$$\begin{aligned}\underline{\tilde{y}_{k+1} \quad &= \quad \tilde{y}_{k+1|k} = y_{k+1} - E[y_{k+1}|\mathbf{y}_k]} \\ &= y_{k+1} - E[H_{k+1}x_{k+1} + v_{k+1}|\mathbf{y}_k] \\ &= y_{k+1} - H_{k+1}\hat{x}_{k+1|k} \\ &= H_{k+1}\tilde{x}_{k+1|k} + v_{k+1}\end{aligned} \tag{16}$$

$$P_{\tilde{y}_{k+1}} = H_{k+1}P_{k+1|k}H_{k+1}^{\mathrm{T}} + R_{k+1} \tag{17}$$

$$\Rightarrow \qquad E\tilde{y}_k = 0, \quad \forall k > 0;$$
$$E\tilde{y}_k\tilde{y}_j^{\mathrm{T}} = 0, \quad \forall k \neq j > 0.$$

结论：新息序列是均值为零的高斯独立随机序列.

3.3 量测修正 (Measurement Updating)

$$\hat{x}_{k+1|k+1} = E[x_{k+1}|\mathbf{y}_{k+1}] = E[x_{k+1}|\mathbf{y}_k, \tilde{y}_{k+1}](\text{此时} < \mathbf{y}_k, \tilde{y}_{k+1} >= 0)$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + \underbrace{\operatorname{cov}[x_{k+1}, \tilde{y}_{k+1}]\operatorname{var}^{-1}[\tilde{y}_{k+1}]}_{K_{k+1}}[\tilde{y}_{k+1} - E\tilde{y}_{k+1}]$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\underbrace{\left[y_{k+1} - H_{k+1}\hat{x}_{k+1|k}\right]}_{\text{new information}} \tag{18}$$

3.4 卡尔曼滤波增益 (Kalman Gain)

$$K_{k+1} = \operatorname{cov}[x_{k+1}, \tilde{y}_{k+1}] \operatorname{var}^{-1}[\tilde{y}_{k+1}]$$

$$\begin{aligned}
\operatorname{cov}[x_{k+1}, \tilde{y}_{k+1}] &= E[x_{k+1} - \bar{x}_{k+1}]\tilde{y}_{k+1}^{\mathrm{T}} \\
&= E[\tilde{x}_{k+1|k} + \hat{x}_{k+1|k} - \bar{x}_{k+1}][H_{k+1}\tilde{x}_{k+1|k} + v_{k+1}]^{\mathrm{T}} \\
&= P_{k+1|k}H_{k+1}^{\mathrm{T}} \\
\operatorname{var}[\tilde{y}_{k+1}] &= H_{k+1}P_{k+1|k}H_{k+1}^{\mathrm{T}} + R_{k+1}
\end{aligned}$$

$$\boxed{K_{k+1} = P_{k+1|k}H_{k+1}^{\mathrm{T}}[H_{k+1}P_{k+1|k}H_{k+1}^{\mathrm{T}} + R_{k+1}]^{-1}} \tag{19}$$

3.5 滤波方差 (Filtering Covariance)

$$
\begin{aligned}
\tilde{x}_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k+1} &= \tilde{x}_{k+1|k} - K_{k+1}[H_{k+1}\tilde{x}_{k+1|k} + v_{k+1}] \\
&= [I - K_{k+1}H_{k+1}]\tilde{x}_{k+1|k} - K_{k+1}v_{k+1}
\end{aligned}
$$

$$\begin{aligned}
\Rightarrow P_{k+1|k+1} &= \underline{[I - K_{k+1}H_{k+1}]P_{k+1|k}[I - K_{k+1}H_{k+1}]^{\mathrm{T}} + K_{k+1}R_{k+1}K_{k+1}^{\mathrm{T}}} \\
&= [I - K_{k+1}H_{k+1}]P_{k+1|k} - P_{k+1|k}H_{k+1}^{\mathrm{T}}K_{k+1}^{\mathrm{T}} \\
&\quad + \underbrace{K_{k+1}H_{k+1}P_{k+1|k}H_{k+1}^{\mathrm{T}}K_{k+1}^{\mathrm{T}} + K_{k+1}R_{k+1}K_{k+1}^{\mathrm{T}}} \\
&= [I - K_{k+1}H_{k+1}]P_{k+1|k} - P_{k+1|k}H_{k+1}^{\mathrm{T}}K_{k+1}^{\mathrm{T}} \\
&\quad + \underbrace{K_{k+1}[H_{k+1}P_{k+1|k}H_{k+1}^{\mathrm{T}} + R_{k+1}]}K_{k+1}^{\mathrm{T}} \\
&= [I - K_{k+1}H_{k+1}]P_{k+1|k} \underbrace{-P_{k+1|k}H_{k+1}^{\mathrm{T}}K_{k+1}^{\mathrm{T}} + P_{k+1|k}H_{k+1}^{\mathrm{T}}K_{k+1}^{\mathrm{T}}}_{=0}
\end{aligned}$$

$$\Rightarrow \boxed{P_{k+1|k+1} = [I - K_{k+1}H_{k+1}]P_{k+1|k}} \tag{20}$$

3.6 初始化 (Initial Filtering)

$$
\begin{aligned}
E[\tilde{x}_{k+1|k+1}] &= E[x_{k+1} - \hat{x}_{k+1|k+1}] \\
&= \Phi_{k+1,k} E x_k - E\{\hat{x}_{k+1|k} + K_{k+1}[y_{k+1} - H_{k+1}\hat{x}_{k+1|k}]\} \\
&= \Phi_{k+1,k} E\tilde{x}_{k|k} - K_{k+1} H_{k+1} \Phi_{k+1,k} E\tilde{x}_{k|k}
\end{aligned}
$$

可以发现，为了保证估计的无偏性，要求

$$
\boxed{
\begin{aligned}
\hat{x}_{0|0} &= E x_0 = \bar{x}_0 = m_0 \qquad (21) \\
P_{0|0} &= \mathrm{var}[x_0] = P_0 \qquad (22)
\end{aligned}
}
$$

3.7 卡尔曼滤波算法

Table 1: 卡尔曼滤波算法

系统方程	$x_{k+1} = \Phi_{k+1,k} x_k + \Psi_{k+1,k} u_k + \Gamma_k w_k$ $y_{k+1} = H_{k+1} x_{k+1} + v_{k+1}$
滤波初值	$\hat{x}_{0\|0} = E x_0 = m_0$ $P_{0\|0} = \mathrm{var}[x_0] = P_0$
一步预测	$\hat{x}_{k+1\|k} = \Phi_{k+1,k} \hat{x}_{k\|k} + \Psi_{k+1,k} u_k$ $P_{k+1\|k} = \Phi_{k+1,k} P_{k\|k} \Phi_{k+1,k}^{\mathrm{T}} + \Gamma_k Q_k \Gamma_k^{\mathrm{T}}$
量测修正	$K_{k+1} = P_{k+1\|k} H_{k+1}^{\mathrm{T}} [H_{k+1} P_{k+1\|k} H_{k+1}^{\mathrm{T}} + R_{k+1}]^{-1}$ $\hat{x}_{k+1\|k+1} = \hat{x}_{k+1\|k} + K_{k+1}[y_{k+1} - H_{k+1} \hat{x}_{k+1\|k}]$ $P_{k+1\|k+1} = [I - K_{k+1} H_{k+1}] P_{k+1\|k}$

3.8 最优滤波等价公式 (Alternative Formula)

将滤波公式整理为

$$\hat{x}_{k+1|k+1} = \underline{[I - K_{k+1}H_{k+1}]}\hat{x}_{k+1|k} + \underline{K_{k+1}}y_{k+1}$$

应用矩阵求逆引理

$$A_1A_2(A_3A_1A_2 + A_4)^{-1} = (A_1^{-1} + A_2A_4^{-1}A_3)^{-1}A_2A_4^{-1}$$

改写卡尔曼滤波增益

$$\begin{aligned} K_{k+1} &= P_{k+1|k}H_{k+1}^{\mathrm{T}}[H_{k+1}P_{k+1|k}H_{k+1}^{\mathrm{T}} + R_{k+1}]^{-1} \\ &= [P_{k+1|k}^{-1} + H_{k+1}^{\mathrm{T}}R_{k+1}^{-1}H_{k+1}]^{-1}H_{k+1}^{\mathrm{T}}R_{k+1}^{-1} \end{aligned} \quad (23)$$

$$
\begin{aligned}
I - K_{k+1}H_{k+1} &= [P_{k+1|k}^{-1} + H_{k+1}^{\mathrm{T}}R_{k+1}^{-1}H_{k+1}]^{-1} \\
&\quad \cdot\{[P_{k+1|k}^{-1} + \underbrace{H_{k+1}^{\mathrm{T}}R_{k+1}^{-1}H_{k+1}] - H_{k+1}^{\mathrm{T}}R_{k+1}^{-1}H_{k+1}}\} \\
&= [P_{k+1|k}^{-1} + H_{k+1}^{\mathrm{T}}R_{k+1}^{-1}H_{k+1}]^{-1}P_{k+1|k}^{-1} \\
P_{k+1|k+1} &= \underline{[I - K_{k+1}H_{k+1}]}P_{k+1|k} \\
&= [P_{k+1|k}^{-1} + H_{k+1}^{\mathrm{T}}R_{k+1}^{-1}H_{k+1}]^{-1}
\end{aligned}
$$

由上式及 (23) 式可得

$$
\boxed{
\begin{aligned}
P_{k+1|k+1} &= [P_{k+1|k}^{-1} + H_{k+1}^{\mathrm{T}}R_{k+1}^{-1}H_{k+1}]^{-1} && (24) \\
K_{k+1} &= P_{k+1|k+1}H_{k+1}^{\mathrm{T}}R_{k+1}^{-1} && (25)
\end{aligned}
}
$$

Table 2: 卡尔曼滤波算法 II

系统方程	$x_{k+1} = \Phi_{k+1,k} x_k + \Psi_{k+1,k} u_k + \Gamma_k w_k$ $y_{k+1} = H_{k+1} x_{k+1} + v_{k+1}$
滤波初值	$\hat{x}_{0\|0} = E x_0 = m_0$ $P_{0\|0} = \mathrm{var}[x_0] = P_0$
时间修正	$\hat{x}_{k+1\|k} = \Phi_{k+1,k} \hat{x}_{k\|k} + \Psi_{k+1,k} u_k$ $P_{k+1\|k} = \Phi_{k+1,k} P_{k\|k} \Phi_{k+1,k}^{\mathrm{T}} + \Gamma_k Q_k \Gamma_k^{\mathrm{T}}$
量测修正	$P_{k+1\|k+1} = [P_{k+1\|k}^{-1} + H_{k+1}^{\mathrm{T}} R_{k+1}^{-1} H_{k+1}]^{-1}$ $K_{k+1} = P_{k+1\|k+1} H_{k+1}^{\mathrm{T}} R_{k+1}^{-1}$ $\hat{x}_{k+1\|k+1} = \hat{x}_{k+1\|k} + K_{k+1}[y_{k+1} - H_{k+1} \hat{x}_{k+1\|k}]$

4. 两个范例

Example 4.1 考虑如下标量线性定常系统:

$$x_{k+1} = ax_k + w_k$$

$$y_{k+1} = x_{k+1} + v_{k+1}$$

其中, a 为常数; $\{w_k\}$ 及 $\{v_k\}$ 均为零均值白噪声序列, 与 x_0 不相关, 而且

$$\underline{Ew_kw_j = q\delta_{kj}, \quad Ev_kv_j = r\delta_{kj}}$$

求 $\hat{x}_{k|k}$ 的递推计算方程。

[Solution] 本问题中, $\Phi_{k+1,k}=a,\Gamma_k=1,u_k=0$, 因此滤波方程如下:

(0) $\hat{x}_{0|0}=\bar{x}_0,P_{0|0}=P_0;$

(1) $\hat{x}_{k+1|k}=a\hat{x}_{k|k};$

(2) $P_{k+1|k}=\Phi_{k+1,k}P_{k|k}\Phi^T_{k+1,k}+\Gamma_kQ_k\Gamma^T_k=a^2P_{k|k}+q;$

(3) $K_{k+1}=P_{k+1|k}H^{\mathrm{T}}_{k+1}[H_{k+1}P_{k+1|k}H^{\mathrm{T}}_{k+1}+R_{k+1}]^{-1}=P_{k+1|k}(P_{k+1|k}+r)^{-1}=\frac{P_{k+1|k}}{P_{k+1|k}+r};$

(4) $\hat{x}_{k+1|k+1}=\hat{x}_{k+1|k}+K_{k+1}[y_{k+1}-H_{k+1}\hat{x}_{k+1|k}]=(1-K_{k+1})\hat{x}_{k+1|k}+K_{k+1}y_{k+1};$

(5) $P_{k+1|k+1}=[I-K_{k+1}H_{k+1}]P_{k+1|k}=(1-K_{k+1})P_{k+1|k}=\frac{rP_{k+1|k}}{P_{k+1|k}+r}=rK_{k+1};$

简单讨论:

♡ 当 $r=0$, $K_{k+1}=1,P_{k+1|k+1}=0,\hat{x}_{k+1|k+1}=y_{k+1}$, 说明量测是完备的;

♡ 当 $r \to +\infty$, $K_{k+1}=0, P_{k+1|k+1}=P_{k+1|k}, \hat{x}_{k+1|k+1}=\hat{x}_{K+1|k}$，说明量测没有带来任何信息;

♡ 一般情况地，$0<K_{k+1}<1$, $P_{k+1|k+1}<P_{k+1|k}$, $P_{k+1|k+1}<r$, 说明滤波精度高于预测精度，滤波精度受限于量测精度;

♡ 不稳定的系统 $(a>1)$ 和建模不确定性 $(q>0)$ 不利于预报，间接影响滤波性能。

Example 4.2 如图1所示，设飞行目标沿视线向雷达作均加速度 $(a=1m/s^2)$ 运动，雷达每一个周期 $(T=1s)$ 测量一次目标相对距离。根据测量数据，求飞行目标相对雷达的距离及相对速度的递推估计。

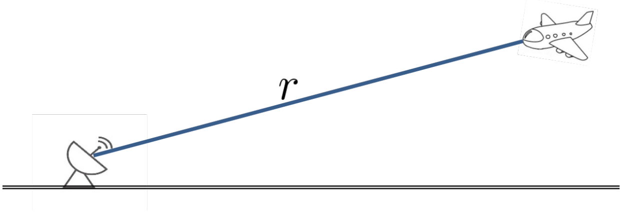

Figure 1: 雷达距离测量示意图

[Solution] 设目标与雷达的相对距离为 r, 那么

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -a$$

令 $x_1 = r, x_2 = \dot{r}$，上述运动方程可以表示为状态方程形式

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a$$

写成向量方程形式

$$\dot{x} = Ax + Bu$$

其中

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad u = a$$

容易求出系统的转移矩阵为

$$\Phi(t,t_0)=e^{A(t-t_0)}=\begin{bmatrix}1 & t-t_0\\ 0 & 1\end{bmatrix}$$

取采样周期 $T=t_{k+1}-t_k=1s$ 进行离散化，可导出

$$x_{k+1}=\Phi_{k+1,k}x_k+\Gamma_k u_k$$

其中

$$x_k=\begin{bmatrix}r_k\\ \dot{r}_k\end{bmatrix},\quad \Phi_{k+1,k}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix},\quad \Gamma_k=\begin{bmatrix}-0.5\\ -1\end{bmatrix},\quad u_k=a$$

由于雷达直接测量相对距离，因此量测方程为

$$y_{k+1}=H_{k+1}x_{k+1}+v_{k+1}$$

其中，$H_{k+1}=[1,0]$。进一步假设

$$Ev_k=0,\quad Ev_k^2=R_k=1.$$

$$\hat{x}_{0|0}=Ex_0=\begin{bmatrix}95\\1\end{bmatrix},\quad P_{0|0}=P_0=\begin{bmatrix}10&0\\0&1\end{bmatrix}.$$

按 $a=1m/s^2$，真实的初始距离和速度分别为 $r_0=100m, \dot{r}_0=0m/s$，可计算出目标不同时刻的距离和速度，见表4。其中，y_k 表示雷达实际测量值。

根据本章介绍的卡尔曼滤波，可递推估计出目标相对雷达的距离和速度，见表4。其中还给出了对应的滤波协方差。

下面给出 $\hat{x}_{1|1}$ 计算过程。

Table 3: 目标实际飞行数据及测量数据

k	0	1	2	3	4	5	6
r_k	100.0	99.5	98.0	95.5	92.0	87.5	82.0
$\dot{r}_k$	0.0	-1.0	-2.0	-3.0	-4.0	-5.0	-6.0
y_k		100.0	97.8	94.4	92.7	87.3	82.1

Table 4: 目标距离及速度滤波数据

k	0	1	2	3	4	5	6
$\hat{r}_k$	95.0	99.6	98.4	95.3	92.6	87.9	82.4
$\hat{\dot{r}}_k$	1.0	0.36	-1.2	-2.3	-3.25	-4.6	-5.7
$P_{11}(k\|k)$	10.0	0.88	0.66	0.66	0.62	0.56	0.50
$P_{22}(k\|k)$	1.0	0.92	0.57	0.30	0.16	0.10	0.06

（1）一步预测：

$$\hat{x}_{1|0} = \Phi_{1|0}\hat{x}_{0|0} + \Gamma_0 u_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 95 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.5 \\ -1.0 \end{bmatrix} = \begin{bmatrix} 95.5 \\ 0 \end{bmatrix}$$

（2）一步预测协方差：$(Q_k = 0, \forall k \geq 0)$

$$P_{1|0} = \Phi_{1|0} P_{0|0} \Phi_{1|0}^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 1 & 1 \end{bmatrix}$$

(3) 卡尔曼滤波增益:

$$
\begin{aligned}
K_1 &= P_{1|0}H_1^T(H_1P_{1|0}H_1^T + R_1)^{-1} \\
&= \begin{bmatrix} 11 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left([1, 0] \begin{bmatrix} 11 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \right)^{-1} \\
&= \begin{bmatrix} 11/12 \\ 1/12 \end{bmatrix} = \begin{bmatrix} 0.92 \\ 0.08 \end{bmatrix}
\end{aligned}
$$

(4) 量测修正:

$$
\hat{x}_{1|1} = \hat{x}_{1|0} + K_1(y_1 - H_1\hat{x}_{1|0}) = \begin{bmatrix} 95.5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.92 \\ 0.08 \end{bmatrix} (100.0 - 95.5) = \begin{bmatrix} 99.64 \\ 0.36 \end{bmatrix}
$$

(5) 滤波协方差:

$$P_{1|1} = (I - K_1H_1)P_{1|0} = \left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}0.92\\ 0.08\end{bmatrix}[1,0]\right)\begin{bmatrix}11 & 1\\ 1 & 1\end{bmatrix}$$
$$= \begin{bmatrix}0.08 & 0\\ -0.08 & 1\end{bmatrix}\begin{bmatrix}11 & 1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}0.88 & 0.10\\ 0.10 & 0.92\end{bmatrix}$$

■

♠ 在实际应用中，在考察 $\hat{x}_i(k|k) \sim k$ 的时候，同时考察对应的 $\pm 3\sqrt{P_{ii}(k|k)} \sim k$ 变化，对于把握状态滤波具有重要的直观意义。

□

5. 相关噪声与成形滤波器

在标准卡尔曼滤波算法中，假设过程噪声、量测噪声都是不相关的随机序列，而且认为两者也是互不相关的. 存在如下三种可能的情况:

- ♠ 过程噪声与量测噪声互相关;
- ♠ 过程噪声是有色噪声;
- ♠ 量测噪声是有色噪声.

此时，需要对标准卡尔曼滤波算法进行必要改造。

5.1 过程噪声与量测噪声互相关

考察状态方程和量测方程

$$
\begin{aligned}
x_{k+1} &= \Phi_{k+1,k} x_k + \Psi_{k+1,k} u_k + \Gamma_k w_k \\
y_{k+1} &= H_{k+1} x_{k+1} + v_{k+1}
\end{aligned}
$$

当过程噪声与量测噪声是互相关随机序列时，这里是指

$$
\boxed{E w_k v_j^{\mathrm{T}} = C_k \delta_{kj}} \tag{26}
$$

式中，$C_k \neq 0$.

问题描述何毓琦（Y C Ho）方法

实施等效变换

$$
\begin{aligned}
x_{k+1} &= \Phi_{k+1,k}x_k + \Psi_{k+1,k}u_k + \Gamma_k w_k + \underline{J_k[y_k - H_k x_k - v_k]} \\
&= \underbrace{[\Phi_{k+1,k} - J_k H_k]}_{\Phi^*_{k+1,k}} x_k + \underbrace{\Psi_{k+1,k}u_k + J_k y_k}_{u^*_k} + \underbrace{\Gamma_k w_k - J_k v_k}_{w^*_k}
\end{aligned}
$$

如果取 $J_k = \Gamma_k C_k R_k^{-1}$, 那么 $E< w^*_k, v_k > = E\{[\Gamma_k w_k - J_k v_k]v_k^{\mathrm{T}}]\} = 0$.

等效系统

$$
\boxed{
\begin{aligned}
x_{k+1} &= \Phi^*_{k+1,k}x_k + u^*_k + w^*_k \\
y_{k+1} &= H_{k+1}x_{k+1} + v_{k+1}
\end{aligned}
}
\begin{aligned}
(27) \\
(28)
\end{aligned}
$$

式中:

$$\Phi^*_{k+1,k} = \Phi_{k+1,k} - J_k H_k$$
$$u^*_k = \Psi_{k+1,k} u_k + J_k y_k$$
$$w^*_k = \Gamma_k w_k - J_k v_k$$

而且

$$Ew^*_k = \Gamma_k Ew_k - J_k Ev_k = 0$$
$$Q^*_k = \mathrm{var}(w^*_k) = \Gamma_k Q_k \Gamma_k^{\mathrm{T}} - J_k C_k^{\mathrm{T}} \Gamma_k^{\mathrm{T}}$$
$$Ew^*_k v_k^{\mathrm{T}} = 0, \quad Ew^*_k x_0^{\mathrm{T}} = 0$$

一步预测相关计算

观察等效系统，可见仅一步预测计算需要修正。

$$
\begin{aligned}
\hat{x}_{k+1|k} &= \Phi^*_{k+1,k}\hat{x}_{k|k} + u^*_k && (29)\\
&= \Phi_{k+1,k}\hat{x}_{k|k} + \Psi_{k+1,k}u_k + J_k[y_k - H_k\hat{x}_{k|k}]\\
P_{k+1|k} &= \Phi^*_{k+1,k}P_{k|k}\Phi^{*\mathrm{T}}_{k+1,k} + Q^*_k && (30)\\
&= [\Phi_{k+1,k} - J_kH_k]P_{k|k}[\Phi_{k+1,k} - J_kH_k]^{\mathrm{T}} + \Gamma_kQ_k\Gamma_k^{\mathrm{T}} - J_kC_k^{\mathrm{T}}\Gamma_k^{\mathrm{T}}
\end{aligned}
$$

如预期的那样，当 $C_k = Ew_kv_k^{\mathrm{T}} = 0$ 时，滤波算法退化为标准卡尔曼滤波算法. 过程噪声与量测噪声互相关时的卡尔曼滤波算法汇总见表5.

Table 5: 过程噪声与量测噪声互相关时的卡尔曼滤波算法

状态方程	$x_{k+1} = \Phi_{k+1,k}x_k + \Psi_{k+1,k}u_k + \Gamma_k w_k$
量测方程	$y_{k+1} = H_{k+1}x_{k+1} + v_{k+1}$
滤波初值	$\hat{x}_{0\|0} = Ex_0 = m_0$
	$P_{0\|0} = \mathrm{var}[x_0] = P_0$
辅助增益	$J_k = \Gamma_k C_k R_k^{-1}$, 其中 $C_k = Ew_k v_k^{\mathrm{T}}$
一步预测	$\hat{x}_{k+1\|k} = \Phi_{k+1,k}\hat{x}_{k\|k} + \Psi_{k+1,k}u_k + J_k[y_k - H_k\hat{x}_{k\|k}]$
	$P_{k+1\|k} = [\Phi_{k+1,k} - J_kH_k]P_{k\|k}[\Phi_{k+1,k} - J_kH_k]^{\mathrm{T}} + \Gamma_kQ_k\Gamma_k^{\mathrm{T}} - J_kC_k^{\mathrm{T}}\Gamma_k^{\mathrm{T}}$
滤波增益	$K_{k+1} = P_{k+1\|k}H_{k+1}^{\mathrm{T}}[H_{k+1}P_{k+1\|k}H_{k+1}^{\mathrm{T}} + R_{k+1}]^{-1}$
滤波计算	$\hat{x}_{k+1\|k+1} = \hat{x}_{k+1\|k} + K_{k+1}[y_{k+1} - H_{k+1}\hat{x}_{k+1\|k}]$
	$P_{k+1\|k+1} = [I - K_{k+1}H_{k+1}]P_{k+1\|k}$

5.2 基于一步预测的滤波算法

将 $\hat{x}_{k|k}$ 带入一步预测

$$\begin{aligned}
\hat{x}_{k+1|k} &= \Phi_{k+1,k}[\hat{x}_{k|k-1} + K_k(y_k - H_k\hat{x}_{k|k-1})] \\
&\quad +\Psi_{k+1,k}u_k + J_k\{y_k - H_k[\hat{x}_{k|k-1} + K_k(y_k - H_k\hat{x}_{k|k-1})]\} \\
&= \Phi_{k+1,k}\hat{x}_{k|k-1} + \Psi_{k+1,k}u_k \\
&\quad + \underbrace{(\Phi_{k+1,k}K_k + J_k - J_kH_kK_k)}_{\triangleq K_k^-}(y_k - H_k\hat{x}_{k|k-1})
\end{aligned}$$

将 K_k 代入修正卡尔曼增益 K_k^-，即

$$\begin{aligned} K_k^- &= \Phi_{k+1,k}K_k + J_k - J_k H_k K_k \\ &= \{\Phi_{k+1,k}P_{k|k-1}H_k^{\mathrm{T}} + J_k[H_k P_{k|k-1}H_k^{\mathrm{T}} + R_k] - J_k H_k P_{k|k-1}H_k^{\mathrm{T}}\} \\ &\quad \times [H_k P_{k|k-1}H_k^{\mathrm{T}} + R_k]^{-1} \\ &= [\Phi_{k+1,k}P_{k|k-1}H_k^{\mathrm{T}} + J_k R_k][H_k P_{k|k-1}H_k^{\mathrm{T}} + R_k]^{-1} \\ &= [\Phi_{k+1,k}P_{k|k-1}H_k^{\mathrm{T}} + \Gamma_k C_k][H_k P_{k|k-1}H_k^{\mathrm{T}} + R_k]^{-1} \end{aligned}$$

一步预测与预测误差

一步预测方程可以改写为

$$\hat{x}_{k+1|k} = \Phi_{k+1,k}\hat{x}_{k|k-1} + \Psi_{k+1,k}u_k + K_k^-(y_k - H_k\hat{x}_{k|k-1}) \tag{31}$$

$$K_k^- = [\Phi_{k+1,k}P_{k|k-1}H_k^{\mathrm{T}} + \Gamma_k C_k][H_k P_{k|k-1}H_k^{\mathrm{T}} + R_k]^{-1} \tag{32}$$

考虑到

$$\begin{aligned}\tilde{x}_{k+1|k} &= x_{k+1} - \hat{x}_{k+1|k} \\ &= \Phi_{k+1,k}x_k + \Psi_{k+1,k}u_k + \Gamma_k w_k \\ &\quad -\Phi_{k+1,k}\hat{x}_{k|k-1} - \Psi_{k+1,k}u_k - K_k^-(y_k - H_k\hat{x}_{k|k-1})\end{aligned}$$

即

$$\tilde{x}_{k+1|k} = [\Phi_{k+1,k} - K_k^- H_k]\tilde{x}_{k|k-1} + \Gamma_k w_k - K_k^- v_k$$

一步预测方差

由于 $E\tilde{x}_{k+1|k} = 0$，因此

$$\begin{aligned}P_{k+1|k} &= E\tilde{x}_{k+1|k}\tilde{x}_{k+1|k}^{\mathrm{T}} \\ &= [\Phi_{k+1,k} - K_k^- H_k]P_{k|k-1}[\Phi_{k+1,k} - K_k^- H_k]^{\mathrm{T}} \\ &\quad +\Gamma_k Q_k \Gamma_k^{\mathrm{T}} + K_k^- R_k K_k^{-\mathrm{T}} - \Gamma_k C_k K_k^{-\mathrm{T}} - K_k^- C_k^{\mathrm{T}} \Gamma_k^{\mathrm{T}}\end{aligned} \tag{33}$$

噪声互相关一步预测滤波算法见表6.

Table 6: 噪声互相关一步预测滤波算法

系统方程	$x_{k+1} = \Phi_{k+1,k} x_k + \Psi_{k+1,k} u_k + \Gamma_k w_k$ $y_{k+1} = H_{k+1} x_{k+1} + v_{k+1}$
滤波初值	$\hat{x}_{0\|0} = E x_0 = m_0$ $P_{0\|0} = \mathrm{var}[x_0] = P_0$
辅助增益	$J_k = \Gamma_k C_k R_k^{-1}$, 其中 $C_k = E w_k v_k^{\mathrm{T}}$
一步预测滤波	$K_k^- = [\Phi_{k+1,k} P_{k\|k-1} H_k^{\mathrm{T}} + \Gamma_k C_k][H_k P_{k\|k-1} H_k^{\mathrm{T}} + R_k]^{-1}$ $\hat{x}_{k+1\|k} = \Phi_{k+1,k} \hat{x}_{k\|k-1} + \Psi_{k+1,k} u_k + K_k^- (y_k - H_k \hat{x}_{k\|k-1})$ $P_{k+1\|k} = [\Phi_{k+1,k} - K_k^- H_k] P_{k\|k-1} [\Phi_{k+1,k} - K_k^- H_k]^{\mathrm{T}}$ $+\Gamma_k Q_k \Gamma_k^{\mathrm{T}} + K_k^- R_k K_k^{-\mathrm{T}} - \Gamma_k C_k K_k^{-\mathrm{T}} - K_k^- C_k^{\mathrm{T}} \Gamma_k^{\mathrm{T}}$
辅助输出	$K_{k+1} = P_{k+1\|k} H_{k+1}^{\mathrm{T}} [H_{k+1} P_{k+1\|k} H_{k+1}^{\mathrm{T}} + R_{k+1}]^{-1}$ $\hat{x}_{k+1\|k+1} = \hat{x}_{k+1\|k} + K_{k+1} [y_{k+1} - H_{k+1} \hat{x}_{k+1\|k}]$ $P_{k+1\|k+1} = [I - K_{k+1} H_{k+1}] P_{k+1\|k}$

5.3 有色过程噪声

系统模型简化与假设

不失一般性，不考虑控制信号的作用，此时系统模型可以描述为

$$x_{k+1} = \Phi_{k+1,k} x_k + \Gamma_k w_k \quad (34)$$

$$y_{k+1} = H_{k+1} x_{k+1} + v_{k+1} \quad (35)$$

假设量测噪声是白噪声，即 $v_{k+1} \sim (0, R)$; $x_0 \sim (\bar{x}_0, P_0)$; x_0, w_k, v_k 是互不相关的.

有色过程噪声是指

$$E w_k w_k^{\mathrm{T}} = Q \geq 0 \quad (36)$$

$$E w_k w_j^{\mathrm{T}} \neq 0, \quad \text{for some } k \neq j \quad (37)$$

一般地，如果噪声序列 w_k 的谱密度 $\Phi_w(\omega)$ 不为常数，那么就称为有色噪声。

成形滤波器（Shaping Filter）

根据谱分解定理，有色噪声序列一般可以看作白噪声序列通过合适的线性定常系统（成形滤波器）的输出.

假设成形滤波器的输入为白噪声 w_k^s, 那么

$$x_{k+1}^s = A^s x_k^s + C^s w_k^s \quad (38)$$

$$w_k = H^s x_k^s + D^s w_k^s \quad (39)$$

有色噪声序列 $\{w_k\}$ 的相关性质完全由成形滤波器参数 $\{A^s, C^s, H^s, D^s\}$ 确定.

等价系统

$$\underbrace{\begin{bmatrix} x_{k+1} \\ x^s_{k+1} \end{bmatrix}}_{x^*_{k+1}} = \underbrace{\begin{bmatrix} \Phi_{k+1,k} & \Gamma_k H^s \\ 0 & A^s \end{bmatrix}}_{\Phi^*_{k+1,k}} \underbrace{\begin{bmatrix} x_k \\ x^s_k \end{bmatrix}}_{x^*_k} + \underbrace{\begin{bmatrix} \Gamma_k D^s \\ C^s \end{bmatrix}}_{\Gamma^*_k} w^s_k \tag{40}$$

$$y_{k+1} = \underbrace{[H_{k+1} 0]}_{H^*_{k+1}} \underbrace{\begin{bmatrix} x_{k+1} \\ x^s_{k+1} \end{bmatrix}}_{x^*_{k+1}} + v_{k+1} \tag{41}$$

滤波算法

采用 (40) 式和 (41) 式中的标注符号，于是可以把等价系统简写为

$$
\begin{aligned}
x^*_{k+1} &= \Phi^*_{k+1,k} x^*_k + \Gamma^*_k w_k \\
y_{k+1} &= H^*_{k+1} x^*_{k+1} + v_{k+1}
\end{aligned}
$$

显然，此时可以采用标准卡尔曼滤波算法，获得扩展状态 $x^*_k = [x_k, x^s_k]^{\mathrm{T}}$ 的最优估计. 这样不仅可以获得系统状态的估计 $\hat{x}_k$，还同时得到了成形滤波器内部状态的估计 $\hat{x}^s_k$.

5.4 有色量测噪声

问题描述

- 仍然考虑方程 (34) 和 (35) 描述的系统, 假设过程噪声是白噪声，即 $w_k \sim (0, Q)$; $x_0 \sim (\bar{x}_0, P_0)$; x_0, w_k, v_k 是互不相关的;
- 假设量测噪声是有色噪声，此时不能采用简单的增加状态维数的方法（为什么?）.

♠ 设量测噪声此时为 n_k, $En_k = 0$, $En_k n_j^{\mathrm{T}} = s_{kj}$. 并假设 n_k 是成形滤波器 $\Phi^s_{k+1,k}$ 的输出，即

$$n_{k+1} = \Phi^s_{k+1,k} n_k + v_k$$

其中 $\{v_k\}$ 是高斯分布随机序列, 而且 $Ev_k = 0, Ev_k v_j^{\mathrm{T}} = r_k \delta_{kj}, \forall k, j$. 同样地，假设 $\{v_k\}$ 与过程噪声 $\{w_k\}$ 不相关.

虚拟量测

$$\begin{aligned} z_k &= y_{k+1} - \Phi^s_{k+1,k} y_k \\ &= H_{k+1} x_{k+1} + n_{k+1} - \Phi^s_{k+1,k} y_k \\ &= H_{k+1}[\Phi_{k+1,k} x_k + \Gamma_k w_k] + n_{k+1} - \Phi^s_{k+1,k}[H_k x_k + n_k] \\ &= [H_{k+1}\Phi_{k+1,k} - \Phi^s_{k+1,k} H_k] x_k + [H_{k+1}\Gamma_k w_k + n_{k+1} - \Phi^s_{k+1,k} n_k] \\ &\triangleq G_k x_k + v_k^* \end{aligned}$$

其中，$v_k^* = H_{k+1}\Gamma_k w_k + v_k$. 显然，$v_k^*$ 是高斯分布随机序列，且 $Ev_k^* = 0$，$Ev_k^* v_j^{*\mathrm{T}} = [H_{k+1}\Gamma_k Q_k \Gamma_k^{\mathrm{T}} H_{k+1}^{\mathrm{T}} + r_k]\delta_{kj}, \forall k, j$. 但 v_k^* 与 w_k 是相关的，且 $Ew_k v_k^{*\mathrm{T}} \triangleq C_k = Q_k \Gamma_k^{\mathrm{T}} H_{k+1}^{\mathrm{T}}$.

等效系统

$$
\begin{aligned}
x_{k+1} &= \Phi_{k+1,k} x_k + \Gamma_k w_k \quad (42) \\
z_k &= G_k x_k + v_k^*, \quad k \geq 0 \quad (43)
\end{aligned}
$$

注意到 z_k 不仅包含了 x_k 的量测信息，还包含了 x_{k+1} 的量测信息．因此，上述等效系统的一步预测值即为我们希望的滤波值．应用过程噪声与测量噪声相关时的滤波算法（一步预测部分），可得

$$\hat{x}_{k+1|k+1} = \Phi_{k+1,k}\hat{x}_{k|k} + K_k(z_k - G_k\hat{x}_{k|k}) \tag{44}$$

$$K_k = [\Phi_{k+1,k}P_{k|k}G_k^{\mathrm{T}} + \Gamma_k C_k][G_k P_{k|k} G_k^{\mathrm{T}} + R_k]^{-1} \tag{45}$$

$$\begin{aligned} P_{k+1|k+1} = {} & [\Phi_{k+1,k} - K_k G_k]P_{k|k}[\Phi_{k+1,k} - K_k G_k]^{\mathrm{T}} \\ & + \Gamma_k Q_k \Gamma_k^{\mathrm{T}} + K_k R_k K_k^{\mathrm{T}} - \Gamma_k C_k K_k^{\mathrm{T}} - K_k C_k^{\mathrm{T}} \Gamma_k^{\mathrm{T}} \end{aligned} \tag{46}$$

初始估计

$$\begin{aligned} z_k &= y_{k+1} - \Phi_{k+1,k}^s y_k \\ G_k &= H_{k+1}\Phi_{k+1,k} - \Phi_{k+1,k}^s H_k \\ R_k &= H_{k+1}\Gamma_k Q_k \Gamma_k^{\mathrm{T}} H_{k+1}^{\mathrm{T}} + r_k \\ C_k &= Q_k \Gamma_k^{\mathrm{T}} H_{k+1}^{\mathrm{T}} \end{aligned}$$

滤波初值

观察 (44) 式可知，在求估计 $\hat{x}_{1|1}$ 时，需要 y_0，即测量是从 0 时刻开始的。因此，我们可以利用该信息来确定初始估计。由静态估计理论（[定理 3-7]）可得

$$\hat{x}_{0|0} = \bar{x}_0 + P_0 H_0^{\mathrm{T}} (H_0 P_0 H_0^{\mathrm{T}} + R_0)^{-1} (y_0 - H_0 \bar{x}_0) \tag{47}$$

$$P_{0|0} = P_0 - P_0 H_0^{\mathrm{T}} (H_0 P_0 H_0^{\mathrm{T}} + R_0)^{-1} H_0 P_0 \tag{48}$$

□

6. 卡尔曼滤波器性能分析

- 在高斯分布统计特性假设下，滤波估计值 $\hat{x}_{k|k}$ 是状态 x_k 的**无偏最小方差估计**，而且 $P_{k|k}$ 就是 x_k 基于测量值 $y_1, y_2, \cdots, y_k$ 的所有估计中最小的均方误差矩阵.
- 卡尔曼滤波算法对非高斯假设亦适用，此时 $\hat{x}_{k|k}$ 是所有线性估计中均方误差最小的无偏最优估计，但不是所有估计中的最优估计.

♠ 考察卡尔曼增益公式

$$K_{k+1} = P_{k+1|k+1} H_{k+1} R_{k+1}^{-1}$$

上式说明，滤波增益“正比于”滤波的不确定性，滤波增益“反比于”量测的不确定性.

♠ 考察状态估计方差矩阵

$$P_{k+1|k} = \Phi_{k+1,k} P_{k|k} \Phi_{k+1,k}^{\mathrm{T}} + \Gamma_k Q_k \Gamma_k^{\mathrm{T}}$$
$$P_{k+1|k+1} = [I - K_{k+1} H_{k+1}] P_{k+1|k}$$

可见，当过程噪声Q_{k+1}较大时，一步预测与滤波精度都会变小. 说明提高模型的精度，将有益于状态的估计.

♠ 再由滤波方差矩阵公式可得

$$P_{k+1|k+1}^{-1} = P_{k+1|k}^{-1} + H_{k+1}^{\mathrm{T}} R_{k+1}^{-1} H_{k+1} \tag{49}$$

上式说明，大的测量噪声 R_{k+1} 将使滤波精度下降。所以，大的测量噪声与大的过程噪声对滤波都是不利的，这和我们的直观感觉是一致的.

♠ 卡尔曼滤波公式算法结构为：

$$\text{滤波值} = \text{一步预测值} + \text{修正项}.$$

这暗喻滤波的精度将高于预测的精度。其实，从 (49) 式也可得出

$$P_{k+1|k+1}^{-1} > P_{k+1|k}^{-1}, \quad \text{即} \quad P_{k+1|k+1} < P_{k+1|k}.$$

♠ 除了一步预测，可以非常容易地建立任意步的预测. 不考虑确定性控制项时，即为

$$\hat{x}_{N|k} = \Phi_{N,k}\hat{x}_{k|k}, \qquad \forall N > k.$$

♠ 综合考虑卡尔曼滤波增益与方差

$$
\begin{aligned}
K_{k+1} &= P_{k+1|k+1} H_{k+1} R_{k+1}^{-1} \\
P_{k+1|k} &= \Phi_{k+1,k} P_{k|k} \Phi_{k+1,k}^{\mathrm{T}} + \Gamma_k Q_k \Gamma_k^{\mathrm{T}} \\
P_{k+1|k+1} &= [I - K_{k+1} H_{k+1}] P_{k+1|k}
\end{aligned}
$$

可见，对于给定的系统，如果事先知道 P_0、Q_k、R_k，那么上述卡尔曼滤波增益与方差可以"离线 (off-line)" 预先计算，从而减小在线计算量。

6.1 稳定性分析

一步预测独立递推算法 将 $\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k[y_k - H_k\hat{x}_{k|k-1}]$ 代入一步预测部分可得:

$$\begin{aligned}
\hat{x}_{k+1|k} &= \Phi_{k+1,k}\hat{x}_{k|k} + \Psi_{k+1,k}u_k \\
&= \Phi_{k+1,k}\{\hat{x}_{k|k-1} + K_k[y_k - H_k\hat{x}_{k|k-1}]\} + \Psi_{k+1,k}u_k \\
&= \Phi_{k+1,k}\hat{x}_{k|k-1} + \Phi_{k+1,k}K_k[y_k - H_k\hat{x}_{k|k-1}] + \Psi_{k+1,k}u_k
\end{aligned}$$

将 $P_{k|k} = [I - K_kH_k]P_{k|k-1}$ 代入一步预测方差有

$$\begin{aligned}
P_{k+1|k} &= \Phi_{k+1,k}P_{k|k}\Phi_{k+1,k}^{\mathrm{T}} + \Gamma_kQ_k\Gamma_k^{\mathrm{T}} \\
&= \Phi_{k+1,k}P_{k|k-1}\Phi_{k+1,k}^{\mathrm{T}} - \Phi_{k+1,k}K_kH_kP_{k|k-1}\Phi_{k+1,k}^{\mathrm{T}} + \Gamma_kQ_k\Gamma_k^{\mathrm{T}}
\end{aligned}$$

基于预测的滤波算法

$$
\begin{aligned}
\hat{x}_{k+1|k} &= \Phi_{k+1,k}\hat{x}_{k|k-1} + \Phi_{k+1,k}K_k[y_k - H_k\hat{x}_{k|k-1}] + \Psi_{k+1,k}u_k \\
P_{k+1|k} &= \Phi_{k+1,k}P_{k|k-1}\Phi_{k+1,k}^{\mathrm{T}} - \Phi_{k+1,k}K_kH_kP_{k|k-1}\Phi_{k+1,k}^{\mathrm{T}} + \Gamma_kQ_k\Gamma_k^{\mathrm{T}} \\
K_k &= P_{k|k-1}H_k^{\mathrm{T}}[H_kP_{k|k-1}H_k^{\mathrm{T}} + R_k]^{-1}
\end{aligned}
$$

上述三个公式可以作为滤波算法单独使用，只要给出初始估计值，就可以递推求出任何时刻的状态最优估计。

采用简化记号: $\hat{x}_{k+1|k} \triangleq \hat{x}_{k+1}$, $P_{k+1|k} \triangleq P_{k+1}$, $\Phi_{k+1,k} \triangleq \Phi_k$, $\Psi_{k+1,k} \triangleq \Psi_k$, 上述可以单独使用的滤波算法即为

$$
\hat{x}_{k+1} = \Phi_k\hat{x}_k + \Phi_kK_k[y_k - H_k\hat{x}_k] + \Psi_ku_k \tag{50}
$$

$$
P_{k+1} = \Phi_kP_k\Phi_k^{\mathrm{T}} - \Phi_kK_kH_kP_k\Phi_k^{\mathrm{T}} + \Gamma_kQ_k\Gamma_k^{\mathrm{T}} \tag{51}
$$

$$
K_k = P_kH_k^{\mathrm{T}}[H_kP_kH_k^{\mathrm{T}} + R_k]^{-1} \tag{52}
$$

误差动力学 采用简化记号后

$$x_{k+1} = \Phi_{k+1,k} x_k + \Psi_{k+1,k} u_k + \Gamma_k w_k = \Phi_k x_k + \Psi_k u_k + \Gamma_k w_k$$

可得

$$\begin{aligned}\tilde{x}_{k+1} &= x_{k+1} - \hat{x}_{k+1} = \Phi_k \tilde{x}_k - \Phi_k K_k [y_k - H_k \hat{x}_k] + \Gamma_k w_k \\ &= \Phi_k [I - K_k H_k] \tilde{x}_k - \Phi_k K_k v_k + \Gamma_k w_k\end{aligned}$$

上式后两项是（随机）输入作用，滤波算法的稳定性只需研究

$$\boxed{\tilde{x}_{k+1} = \Phi_k [I - K_k H_k] \tilde{x}_k} \tag{53}$$

稳定性分析 根据 Lyapunov 稳定性理论，取 $V(\tilde{x}_k) = \tilde{x}_k^{\mathrm{T}} P_k^{-1} \tilde{x}_k$. 欲使

$$\underline{\Delta V(\tilde{x}_k) = V(\tilde{x}_{k+1}) - V(\tilde{x}_k) < 0} \tag{54}$$

$$
\begin{aligned}
\Leftrightarrow \quad & \tilde{x}_k^{\mathrm{T}}\{[I-K_kH_k]^{\mathrm{T}}\Phi_k^{\mathrm{T}}P_{k+1}^{-1}\Phi_k[I-K_kH_k]-P_k^{-1}\}\tilde{x}_k<0 \\
\Leftrightarrow \quad & \underline{[I-K_kH_k]^{\mathrm{T}}\Phi_k^{\mathrm{T}}}P_{k+1}^{-1}\underline{\Phi_k[I-K_kH_k]}-P_k^{-1}<0 \\
\Leftrightarrow \quad & P_{k+1}^{-1}-\Phi_k^{-\mathrm{T}}[I-K_kH_k]^{-\mathrm{T}}P_k^{-1}[I-K_kH_k]^{-1}\Phi_k^{-1}<0 \\
\Leftrightarrow \quad & I-P_{k+1}\Phi_k^{-\mathrm{T}}[I-K_kH_k]^{-\mathrm{T}}P_k^{-1}[I-K_kH_k]^{-1}\Phi_k^{-1}<0
\end{aligned}
$$

注意到

$$
\begin{aligned}
P_{k+1} &= \Phi_kP_k\Phi_k^{\mathrm{T}}-\Phi_kK_kH_kP_k\Phi_k^{\mathrm{T}}+\Gamma_kQ_k\Gamma_k^{\mathrm{T}} \\
&= \Phi_k[I-K_kH_k]P_k\Phi_k^{\mathrm{T}}+\Gamma_kQ_k\Gamma_k^{\mathrm{T}} \\
&= \Phi_k[I-K_kH_k]P_k[I-K_kH_k]^{\mathrm{T}}\Phi_k^{\mathrm{T}} \\
&\quad +\Phi_k[I-K_kH_k]P_kH_k^{\mathrm{T}}K_k^{\mathrm{T}}\Phi_k^{\mathrm{T}}+\Gamma_kQ_k\Gamma_k^{\mathrm{T}}
\end{aligned}
$$

又由 $K_k = P_k H_k^{\mathrm{T}}[H_k P_k H_k^{\mathrm{T}} + R_k]^{-1}$ 可知

$$
\begin{aligned}
& K_k[H_k P_k H_k^{\mathrm{T}} + R_k] = P_k H_k^{\mathrm{T}} \\
\Rightarrow \quad & K_k H_k P_k H_k^{\mathrm{T}} = P_k H_k^{\mathrm{T}} - K_k R_k \\
\Rightarrow \quad & (I - K_k H_k) P_k H_k^{\mathrm{T}} = P_k H_k^{\mathrm{T}} - K_k H_k P_k H_k^{\mathrm{T}} = K_k R_k \\
\Rightarrow \quad & P_{k+1} = \Phi_k [I - K_k H_k] P_k [I - K_k H_k]^{\mathrm{T}} \Phi_k^{\mathrm{T}} \\
& + \Phi_k K_k R_k K_k^{\mathrm{T}} \Phi_k^{\mathrm{T}} + \Gamma_k Q_k \Gamma_k^{\mathrm{T}}
\end{aligned}
$$

$$
\begin{aligned}
\Delta V(\tilde{x}_k) \quad = \quad & V(\tilde{x}_{k+1}) - V(\tilde{x}_k) < 0 \\
\Leftrightarrow \quad & -[\Phi_k K_k R_k K_k^{\mathrm{T}} \Phi_k^{\mathrm{T}} + \Gamma_k Q_k \Gamma_k^{\mathrm{T}}] \\
& \cdot \Phi_k^{-\mathrm{T}} [I - K_k H_k]^{-\mathrm{T}} P_k^{-1} [I - K_k H_k]^{-1} \Phi_k^{-1} < 0 \\
\Leftrightarrow \quad & -[\Phi_k K_k R_k K_k^{\mathrm{T}} \Phi_k^{\mathrm{T}} + \Gamma_k Q_k \Gamma_k^{\mathrm{T}}] < 0
\end{aligned}
$$

所以，当 $\Gamma_k Q_k \Gamma_k^{\mathrm{T}} > 0$，或者当 $R_k > 0, Q_k \geq 0$，另外 $K_k \neq 0, \Phi_k$ 可逆，则有 $\Delta V(\tilde{x}_k) < 0$.

6.2 稳态性能

滤波误差来源 设滤波实际的初值有误差

$$\bar{x}_0^* \neq \bar{x}_0 \tag{55}$$

$$P_0^* \neq P_0 \tag{56}$$

滤波过程中的统计特性也不准确

$$Q_k^* \neq Q_k \tag{57}$$

$$R_k^* \neq R_k \tag{58}$$

那么实际计算出的滤波值 $\hat{x}^*_{k|k}$ 与滤波方差矩阵 $P^*_{k|k}$ 分别称为滤波的视在值及视在方差。而实际的方差为

$$P^{**}_{k|k} = \mathrm{var}(x_k - \hat{x}^*_k) = (I - K^*_k H_k) P^{**}_{k|k-1} (I - K^*_k H_k)^T \tag{59}$$

我们将面临三个方差矩阵：$P_{k|k}$、$P^*_{k|k}$ 和 $P^{**}_{k|k}$。一般无法得到最优的 $P_{k|k}$ 与实际的 $P^{**}_{k|k}$，但肯定有 $P^{**}_{k|k} \geq P_{k|k}$。

根据线性系统理论，可以建立如下主要结论：

Theorem 6.1 渐进稳定的卡尔曼滤波器最终趋于无偏估计，即

$$\lim_{k\to+\infty} E\tilde{x}^*_{k|k} = 0.$$

Theorem 6.2 若卡尔曼滤波器是渐进稳定的，且仅仅是初始方差矩阵有误差，即 $P_0^* \neq P_0$，那么

$$\lim_{k\to+\infty} P^*_{k|k} = P_{k|k}.$$

Theorem 6.3 若过程噪声和测量噪声都是平稳的，即

$$Q_k = Q, \quad R_k = R$$

所研究的系统还是定常的，即

$$\Phi_{k+1,k} = \Phi, \quad \Gamma_{k+1,k} = \Gamma, \quad H_k = H$$

当卡尔曼滤波器是渐进稳定的时，有

$$\lim_{k\to+\infty} P_{k|k} = \lim_{k\to+\infty} P_{k|k}^{*} = P \tag{60}$$

$$\lim_{k\to+\infty} K_{k|k} = \lim_{k\to+\infty} K_{k|k}^{*} = K \tag{61}$$

Theorem 6.4 (保守滤波器) 当 $P_0^* \geq P_0$, 且

$$Q_k^* \geq Q_k, \quad R_k^* \geq R_k, \quad \forall k$$

那么

$$\boxed{P_{k|k}^* \geq P_{k|k}^{**}.}$$

□

Questions?